Technical Report 2006-1-001

On Doubly Coprime Factorizability of Diagonal Matrix consisting of Anantharam's Example

Kazuyoshi MORI

February 23, 2006

Department of Computer Software
The University of Aizu
Aizu-Wakamatsu 965-8580, Japan

Technical Report 2006-1-001

Title:
On Doubly Coprime Factorizability of Diagonal Matrix consisting of Anantharam's Example

Authors:
Kazuyoshi MORI

Key Words and Phrases:
Linear systems, Feedback stabilization, Coprime factorization over commutative rings

Abstract:

It is known that Anantharam showed an example of a stabilizable plant which does not admit a doubly coprime factorization. In this report, we show (i) that a diagonal matrix consisting of its plant with even size admits a doubly coprime factorization and (ii) that a diagonal matrix consisting of its plant with odd size does not admit a doubly coprime factorization.

Report Date:
February 23, 2006

Written Language:
English

Any Other Identifying Information of this Report:
Part of manuscript submitted for publication

Distribution Statement:

Supplementary Notes:

Information Systems Laboratory
The University of Aizu
Aizu-Wakamatsu 965-8580, Japan

On Doubly Coprime Factorizability of Diagonal Matrix consisting of Anantharam's Example

Kazuyoshi MORI†

†School of Computer Science and Engineering
The University of Aizu
Aizu-Wakamatsu 965-8580, JAPAN
Tel: +81.242.37.2615, Fax: +81.242.37.2747
Email: Kazuyoshi.MORI@IEEE.ORG

February 23, 2006

Abstract. It is known that Anantharam showed an example of a stabilizable plant which does not admit a doubly coprime factorization. In this report, we show (i) that a diagonal matrix consisting of its plant with even size admits a doubly coprime factorization and (ii) that a diagonal matrix consisting of its plant with odd size does not admit a doubly coprime factorization.

Key words. Linear systems, Feedback stabilization, Coprime factorization over commutative rings

1. Introduction. In this note we are concerned with the coordinate-free approach to control systems. This approach is a factorization approach but does not require the coprime factorizations of the plant.

It is known that Anantharam[1] showed an example of a stabilizable plant which does not admit a doubly coprime factorization. In this report, we show (i) that a diagonal matrix consisting of its plant with even size admits a doubly coprime factorization and (ii) that a diagonal matrix consisting of its plant with odd size does not admit a doubly coprime factorization.

2. Preliminaries. In the following we begin by introducing notations used in this note. Then we give the formulation of the feedback stabilization problem.

2.1. Notations.

Commutative Rings. We will consider that *the set of all stable causal transfer functions* is a commutative ring, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \text{ is a nonzerodivisor}\}$. This will be considered to be *the set of all possible transfer functions*. If the commutative ring $\mathcal{A}$ is an integral domain, $\mathcal{F}$ becomes a field of fractions of $\mathcal{A}$. However, if $\mathcal{A}$ is not an integral domain, then $\mathcal{F}$ is not a field, because any zerodivisor of $\mathcal{F}$ is not a unit.

Matrices. Suppose that x and y denote sizes of matrices.

The set of matrices over $\mathcal{A}$ of size $x \times y$ is denoted by $\mathcal{A}^{x\times y}$. In particular, the set of square matrices over $\mathcal{A}$ of size x is denoted by $(\mathcal{A})_x$. The identity and the zero matrices are denoted by E_x and $O_{x\times y}$, respectively, if the sizes are required, otherwise they are denoted simply by E and O.

Matrices A and B over $\mathcal{A}$ are *right-coprime over* $\mathcal{A}$ if there exist matrices $\widetilde{X}$ and $\widetilde{Y}$ over $\mathcal{A}$ such that $\widetilde{X}A + \widetilde{Y}B = E$. Analogously, matrices $\widetilde{A}$ and $\widetilde{B}$ over $\mathcal{A}$ are *left-coprime over* $\mathcal{A}$ if there exist matrices X and Y over $\mathcal{A}$ such that $\widetilde{A}X + \widetilde{B}Y = E$. Further, pair (N, D) of matrices N and D is said to be a *right-coprime factorization of* P *over* $\mathcal{A}$ if (i) the matrix D is nonsingular over $\mathcal{A}$, (ii) $P = ND^{-1}$ over $\mathcal{F}$, and (iii) N and D are right-coprime over $\mathcal{A}$. Also, pair $(\widetilde{N}, \widetilde{D})$ of matrices $\widetilde{N}$ and $\widetilde{D}$ is said to be a *left-coprime factorization of* P *over* $\mathcal{A}$ if (i) $\widetilde{D}$ is nonsingular over $\mathcal{A}$, (ii) $P = \widetilde{D}^{-1}\widetilde{N}$ over $\mathcal{F}$, and (iii) $\widetilde{N}$ and $\widetilde{D}$ are left-coprime over $\mathcal{A}$. As we have seen, in the case where a matrix is potentially used to express *left* fractional form and/or *left* coprimeness, we usually attach a tilde '$\widetilde{\ }$' to a symbol; for example $\widetilde{N}$, $\widetilde{D}$ for $P = \widetilde{D}^{-1}\widetilde{N}$ and $\widetilde{Y}$, $\widetilde{X}$ for

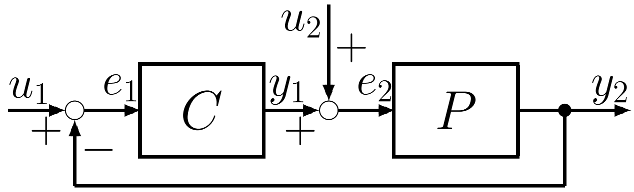

FIG. 2.1. *Feedback system* Σ.

$\widetilde{Y}N + \widetilde{X}D = E$.

2.2. Feedback Stabilization Problem. The stabilization problem considered in this note follows that of Sule in [2] and Mori and Abe in [3] who consider the feedback system Σ [4, Ch.5, Figure 5.1] as in Figure 2.1. For further details the reader is referred to [4, 3]. Throughout this note, the plant we consider has m inputs and n outputs, and its transfer matrix, which itself is also called simply a *plant*, is denoted by P and belongs to $\mathcal{F}^{n\times m}$.

DEFINITION 2.1. *Define* $\widehat{F}_{\rm ad}$ *by*

$$\widehat{F}_{\rm ad} = \{(X,Y) \in \mathcal{F}^{x\times y} \times \mathcal{F}^{y\times x} \mid \ \det(E_x + XY) \textit{ is a unit of } \mathcal{F}, \\ x \textit{ and } y \textit{ are positive integers}\}.$$

For $P \in \mathcal{F}^{n\times m}$ *and* $C \in \mathcal{F}^{m\times n}$*, the matrix* $H(P,C) \in (\mathcal{F})_{m+n}$ *is defined by*

$$H(P,C) = \begin{bmatrix} (E_n + PC)^{-1} & -P(E_m + CP)^{-1} \\ C(E_n + PC)^{-1} & (E_m + CP)^{-1} \end{bmatrix} \tag{2.1}$$

provided $(P,C) \in \widehat{F}_{\rm ad}$. *This* $H(P,C)$ *is the transfer matrix from* $[\,u_1^t \quad u_2^t\,]^t$ *to* $[\,e_1^t \quad e_2^t\,]^t$ *of the feedback system* Σ. *If (i)* $(P,C) \in \widehat{F}_{\rm ad}$ *and (ii)* $H(P,C) \in (\mathcal{A})_{m+n}$, *then we say that the plant* P *is* stabilizable, P *is* stabilized *by* C, *and* C *is a* stabilizing controller *of* P.

Here we define the causality of transfer functions, which is an important physical constraint, used in this note. We employ the definition of causality from Vidyasagar *et al.*[5, Definition 3.1] and Mori and Abe[3].

DEFINITION 2.2. *Let* $\mathcal{Z}$ *be a prime ideal of* $\mathcal{A}$, *with* $\mathcal{Z} \neq \mathcal{A}$, *including all zerodi-*

visors. Define the subsets $\mathcal{P}$ and $\mathcal{P_S}$ of $\mathcal{F}$ as follows:

$$\mathcal{P} = \{n/d \in \mathcal{F} \,|\, n \in \mathcal{A},\ d \in \mathcal{A}\backslash\mathcal{Z}\},\ \ \mathcal{P_S} = \{n/d \in \mathcal{F} \,|\, n \in \mathcal{Z},\ d \in \mathcal{A}\backslash\mathcal{Z}\}.$$

A transfer function in $\mathcal{P}$ $\left(\mathcal{P_S}\right)$ is called causal $\left(\text{strictly causal}\right)$. *Similarly, if every entry of a transfer matrix over $\mathcal{F}$ is in $\mathcal{P}$ $\left(\mathcal{P_S}\right)$, the transfer matrix is called* causal $\left(\text{strictly causal}\right)$.

It should be noted that when using "a stabilizing controller," we do not guarantee the causality. However, in the classical case of the factorization approach, once we restrict ourselves to strictly proper plants, it is known that any stabilizing controller of strictly causal plant is causal (cf. Corollary 5.2.20 of [4], Theorem 4.1 of [5], and Proposition 6.2 of [3]). One can see, in fact, that many practical systems are strictly causal. On the other hand, including noncausal stabilizing controllers seems to make the theory easy and simple in the mathematical viewpoint. From these observations, we have accepted the possibility of the noncausality of stabilizing controllers.

NOTE 2.1. *Let P be a plant and C a stabilizing controller. Suppose that P and C admit a doubly coprime factorization as follows:*

$$\widetilde{Y}N + \widetilde{X}D = I, \tag{2.2}$$

where N, D, $\widetilde{Y}$, $\widetilde{X}$ are matrices over $\mathcal{A}$ with $P = ND^{-1}$ and $C = \widetilde{X}^{-1}\widetilde{Y}$.

NOTE 2.2. *([4, 5.1.32]) Let P be a plant and C a stabilizing controller. Suppose that $(\widetilde{X}, \widetilde{Y})$ be a left-coprime factorization of C. Let*

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = H(P, C).$$

Then $(-H_{12}\widetilde{X}^{-1}, -H_{22}\widetilde{X}^{-1})$ is a right-coprime factorization of P.

3. Main Result. In this section, we consider that $\mathcal{A} = \mathbb{Z}\sqrt{-5}$ and $p = (1 + \sqrt{-5})/2$.

Our main results can be stated as follows.

THEOREM 3.1. *The plant* $\mathrm{Diag}(p, p)$ *admits a doubly coprime factorization.*

THEOREM 3.2. *The plant* $\mathrm{Diag}(p, p, p)$ *does not admit a doubly coprime factorization.*

Proof of Theorem 3.1. The work we need is to present an example.

Let $P = \mathrm{Diag}(p, p)$. Thus

$$P = \mathrm{Diag}((1+\sqrt{-5})/2, (1+\sqrt{-5})/2) = \begin{bmatrix} (1+\sqrt{-5})/2 & 0 \\ 0 & (1+\sqrt{-5})/2 \end{bmatrix}.$$

Let

$$\begin{aligned} N = \widetilde{N} &= \begin{bmatrix} 1+\sqrt{-5} & 2-\sqrt{-5} \\ 2-\sqrt{-5} & -3 \end{bmatrix}, \\ D = \widetilde{D} &= \begin{bmatrix} 2 & -1-\sqrt{-5} \\ -1-\sqrt{-5} & -1+\sqrt{-5} \end{bmatrix}, \\ Y &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \widetilde{Y} = \begin{bmatrix} 3+2\sqrt{-5} & 3-2\sqrt{-5} \\ 5 & -2-2\sqrt{-5} \end{bmatrix}, \\ X &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{X} = \begin{bmatrix} 3-2\sqrt{-5} & -6 \\ -2-2\sqrt{-5} & -3+2\sqrt{-5} \end{bmatrix}. \end{aligned}$$

Then we see that $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{Y}$ and

$$\begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} \begin{bmatrix} D & -Y \\ N & X \end{bmatrix} = I.$$

Thus, P admits a doubly coprime factorization. □

Note that from this proof, we see that a stabilizing controller is given as

$$(\widetilde{X}^{-1}\widetilde{Y} =)\ YX^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

This matrix is obviously over $\mathcal{A}$. Hence the plant $\mathrm{Diag}(p, p)$ is strongly stabilizable.

Proof of Theorem 3.2. Let $P = \mathrm{Diag}(p, p, p)$.

Let N and D be

$$\begin{bmatrix} 1+\sqrt{-5} & 2-\sqrt{-5} & 0 \\ 2-\sqrt{-5} & -3 & 0 \\ 0 & 0 & 1+\sqrt{-5} \end{bmatrix} \text{ and}$$
$$\begin{bmatrix} 2 & -1-\sqrt{-5} & 0 \\ -1-\sqrt{-5} & -1+\sqrt{-5} & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

respectively. We then see that $P = ND^{-1}$ holds. Note that the submatrices of the first two rows and columns of N and D here are equal to N and D, respectively, of the proof of Theorem 3.1.

Now let T be $[N^t\ D^t]^t$, that is,

$$T = \begin{bmatrix} N \\ D \end{bmatrix} = \begin{bmatrix} 1+\sqrt{-5} & 2-\sqrt{-5} & 0 \\ 2-\sqrt{-5} & -3 & 0 \\ 0 & 0 & 1+\sqrt{-5} \\ 2 & -1-\sqrt{-5} & 0 \\ -1-\sqrt{-5} & -1+\sqrt{-5} & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{3.1}$$

Let I be the ideal generated by the all full-size minors of T.

Let T' be the first two columns of the matrix T, that is,

$$T' = \begin{bmatrix} 1+\sqrt{-5} & 2-\sqrt{-5} \\ 2-\sqrt{-5} & -3 \\ 0 & 0 \\ 2 & -1-\sqrt{-5} \\ -1-\sqrt{-5} & -1+\sqrt{-5} \\ 0 & 0 \end{bmatrix}. \tag{3.2}$$

From the proof of Theorem 3.1, we know that the ideal generated by the all full-size minors of T' is equal to $\mathcal{A}$. It follows that the ideal I is generated by the $(3,3)$- and the $(6,3)$-entries of T. That is, the ideal I is equal to $(1+\sqrt{-5}, 2)$. It is known that this is not equal to $\mathcal{A}$ (see [1]).

Thus the plant $\mathrm{Diag}(p,p,p)$ does not admit a doubly coprime factorization. □

From the main results above, we have the following corollary.

COROLLARY 3.3. *Let* $\mathcal{A} = \mathbb{Z}\sqrt{-5}$ *and* $p = (1+\sqrt{-5})/2$. *Then the plant* $\mathrm{Diag}(p,p,\ldots,p)$ *with even size admits a doubly coprime factorization. On the other hand, the plant* $\mathrm{Diag}(p,p,\ldots,p)$ *with odd size does not admit a doubly coprime factorization.*

NOTE 3.1. *Let* $\mathcal{A}$ *be a commutative ring and* p *a plant. Suppose that* $\mathrm{Diag}(p,p)$ *admits a doubly coprime factorization. Even so, we cannot determine whether* p *admits a doubly coprime factorization or not. It depends on the commutative ring* $\mathcal{A}$.

REFERENCES

[1] V. Anantharam, "On stabilization and the existence of coprime factorizations," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1030–1031, Oct. 1985.

[2] V. R. Sule, "Feedback stabilization over commutative rings: The matrix case," *SIAM J. Control and Optim.*, vol. 32, no. 6, pp. 1675–1695, 1994.

[3] K. Mori and K. Abe, "Feedback stabilization over commutative rings: Further study of coordinate-free approach," *SIAM J. Control and Optim.*, vol. 39, no. 6, pp. 1952–1973, 2001.

[4] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, Cambridge, MA: MIT Press, 1985.

[5] M. Vidyasagar, H. Schneider, and B. A. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 880–894, Apr. 1982.